Real Analysis II

Department of Mathematics and Computer Applications

Year Three

Chapter Three ⇒ Metric Spaces (Bounded sets-Cmpact sets)

Chapter Four ⇒ Continuous Functions.

Chapter Five ⇒ Sequence and Series of Functions.

Chapter six ⇒ Riemann Integral.

Chapter seven ⇒ Differentiation Real-Valued Functions

Definition 0.0.1 (Bounded sets)**.**
Let (X, d) be a metric space. A non-empty S subset of X is said to be **bounded** if for every $x_0 \in S$, there exists $r > 0$ such that

$$d(x, x_0) < r \quad \text{for all } x \in S.$$

In other words S is bounded if there exists $r > 0$ such that $S \subseteq B(x, r)$ for some $x \in S$. *Geometrically* this means that S is bounded if and only if there exists a ball with centre x_0 contains S.

Definition 0.0.2 (Diameter of a set)**.**
Let (X, d) be a metric space and S be a non-empty subset of X. Set

$$\hat{S} = \{d(x, y) : x, y \in S\}$$

$\hat{S}$ is a subset of $\mathbb{R}$ and it is bounded below by 0. If S is bounded then $\hat{S}$ is bounded above and

$$D(S) = \text{diam}(S) = \sup(\hat{S})$$

is called the **diameter** of S

Lemma 0.0.3. *Let (X, d) be a metric space. A non-empty subset S of X is bounded if and only if $\hat{S}$ is bounded above.*

Proof. Homework □

Example 0.0.4.

(1) Let X be the euclidean metric space $\mathbb{R}$. Then $\mathbb{R}$ and $\mathbb{Q}$ are unbounded set.
For $a, b \in \mathbb{R}$, the sets $[a, b], (a, b)$ are bounded and the diameter for each one is $b - a$. Why?

(2) Let X be the euclidean metric space $\mathbb{R}^n$. For $r > 0$ and $x_0 \in \mathbb{R}^n$ let $S = B(x_0, r)$. Then S is bounded and its diameter is $2r$. Why?
Note that for any $x, y \in B(x_0, r)$ we have $d(x, y) \leq d(x, x_0) + d(x_0, y) < r + r = 2r$. Thus

$$\hat{S} = \{d(x, y) : x, y \in B(x_0, r)\} < 2r$$

and $\sup(\hat{S}) = 2r$. Hence $\text{diam}(B(x_0, r))$ is $2r$.

Let $D(x_0, r)$ be a disk in $\mathbb{R}^n$. Show that $D(x_0, r)$ is bounded set and find its diameter.

(3) Let X be any non-empty set. The discrete metric on X defined by

$$d(x, y) := \begin{cases} 0, & \text{when } x = y, \\ 1, & \text{when } x \neq y. \end{cases}$$

Let $x_0 \in X$ and

$$S = \left\{ x \in X : d(x, x_0) < \frac{1}{2} \right\}.$$

Note that S has only one point x_0, thus S is bounded. Moreover,

$$\hat{S} = \{d(x, y) : x, y \in S\} = 0$$

hence $\sup(S) = 0$ and $\text{diam}(S) = 0$.

Theorem 0.0.5 (Cantor's Nested Sets Theorem)**.**

Let (X, d) be a metric space. Suppose that $\{E_n\}_{n=1}^{\infty}$ is a sequence of bounded subset of X such that

(1) *$E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots \supseteq E_n \supseteq \cdots$*

(2) *E_n is non-empty and closed for all $n \in \mathbb{N}$*

(3) *$D(E_n) = diam(E_n) \to 0$ as $n \to \infty$.*

If (X, d) is a complete metric space then

$$\bigcap_n E_n \neq \phi,$$

that is, it is not empty and contains only one point.

Contraction Mapping

Definition 0.0.6. Let (X, d) be a metric space. A **fixed point** of a mapping $T : X \to X$ of a set X into itself is a point $x_0 \in X$ which is mapped onto itself (is "kept fixed" by T), that is,

$$Tx_0 = x_0$$

the image Tx_0 coincides with x_0.

Example 0.0.7. Let X be the euclidean metric space $\mathbb{R}$. suppose that $T : \mathbb{R} \to \mathbb{R}$ be a function defined by $T(x) = x^2$. Then T has two fixed point 0 and 1.

Definition 0.0.8. Let (X, d) be a metric space. A mapping $T : X \to X$ is called a **contraction mapping** on X if there is a real number $0 \leq \alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Geometrically this means that any points x and y have images that are closer together than those points x and y.

Example 0.0.9. Let X be the euclidean metric space $\mathbb{R}$. Then $T : \mathbb{R} \to \mathbb{R}$ such that $T(x) = \frac{x}{2}$, then T is a contraction mapping.

Solution: Let $x, y \in \mathbb{R}$, then

$$\begin{aligned} d(T(x), T(y)) &= d\left(\frac{x}{2}, \frac{y}{2}\right) \\ &= \left|\frac{x}{2} - \frac{y}{2}\right| \\ &= \frac{1}{2}|x - y| \\ &= \frac{1}{2}d(x, y), \end{aligned}$$

thus $d(Tx, Ty) = \alpha d(x, y)$, where $\alpha = 0.5$.

Theorem 0.0.10 (Fixed point theorem)**.**
Let (X, d) be a complete metric space. If $T : X \to X$ is a contraction mapping then there exists only one point $x_0 \in X$ such that:

$$Tx_0 = x_0,$$

that is the equation $Tx_0 = x_0$ has only one solution.

3.4 Compact Metric Spaces

Definition 0.0.11. Let (X, d) be a metric space. An **open cover** of S in X is a collection of open sets $\{G_\alpha\}$ whose union "covers" (contains) S. That is

$$S \subseteq \bigcup_\alpha G_\alpha.$$

A **subcover** of $\{G_\alpha\}$ is a subcollection $\{G_{\alpha_i}\}$ that still covers S.

Example 0.0.12. Let $S = [\frac{1}{2}, 1)$ in $\mathbb{R}$. Then S has a cover $\{G_n\}_{n=3}^{\infty}$ where $G_n = (\frac{1}{n}, 1 - \frac{1}{n})$.

$G_3 = (\frac{1}{3}, 1 - \frac{1}{3}) = (\frac{1}{3}, \frac{2}{3})$

$G_4 = (\frac{1}{4}, 1 - \frac{1}{4}) = (\frac{1}{4}, \frac{3}{4})$

$G_5 = (\frac{1}{5}, 1 - \frac{1}{5}) = (\frac{1}{5}, \frac{4}{5})$

:

:

$G_{10} = (\frac{1}{10}, 1 - \frac{1}{10}) = (\frac{1}{10}, \frac{9}{10})$

:

:

Also, $\{(0, 2)\}$ is an open cover of S contains one element.

Also, $\{W_x\}_{x \in [\frac{1}{2}, 1)}$, where $W_x = B(x, 0.1)$ is an open cover for S.

In each cover, do we need all these sets to still cover S?

Note that:

$\{G_n\}_{n=3}^{\infty}$ has finite subcover $\{G_n\}_{n=122}^{\infty}$ and is still cover the set S.

$\{(0, 2)\}$ has finite subcover $\{(0, 2)\}$ and this still covers S.

$\{W_x\}_{x \in [\frac{1}{2}, 1)}$ has finite subcover $\left\{W_{\frac{5}{10}}, W_{\frac{6}{10}}, W_{\frac{7}{10}}, W_{\frac{8}{10}}, W_{\frac{9}{10}}\right\}$ and this still covers S.

Example 0.0.13. Let $S = [0, 1]$ in $\mathbb{R}$. Then S has a cover $\{G_n\}_{n=3}^{\infty} \cup \{W_0, W_1\}$. This cover has a finite subcover $\{W_0, W_1, G_{11}\}$, and this still covers S.

Definition 0.0.14. Let (X, d) be a metric space. A set S is **compact** in X, if for every open cover of S contains a finite subcover.

*So, S is **not compact** means, there exists some open cover that has no finite subcover.*

If $S = X$ then we say that X is a **compact space**.

Example 0.0.15. $S = (0, 1)$ in $\mathbb{R}$ is not compact. We can not find a finite subcover for $\{G_n\}_{n=3}^{\infty}$, that still cover S.

Example 0.0.16. $\mathbb{Z}$ in $\mathbb{R}$ is not compact.

Homework

Example 0.0.17. Show that $\left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$ is not compact in $\mathbb{R}$.

Homework

Example 0.0.18. $[0,1]$ in $\mathbb{R}$, may be compact? we need to check every open cover? (or prove theorem).

Lemma 0.0.19. *Finite sets are compact.*

Proof. Let (X,d) be a metric space and $S \subseteq X$ be a finite set. Consider an open cover $\{G_\alpha\}$ covering $S = \{x_1, x_2, \cdots x_n\}$.

For all x_i choose one G_{α_i} that contains x_i. Then

$$\{G_{\alpha_i}\}_{i=1}^n$$

covers the set S and it is finite subcover from $\{G_\alpha\}$. □

Theorem 0.0.20. *Compact sets are bounded.*

Proof. Let (X,d) be a metric space and S be a compact set in X.

Consider B_x be a ball with radius 1, that is $B_x = B(x,1)$, then $\{B_x\}_{x\in S}$ is an open cover for S.

By compactness of S there exists a finite subcover $\{B_{x_i}\}_{i=1}^n$. Set

$$R = \max_{1\leq i,j\leq n} \{d(x_i, x_j)\}$$

this maximum exists because the set $\{x_1, x_2, \cdots, x_n\}$ is finite. Then $B(x_1, R+2)$ contains S, hence S is bounded. □

Theorem 0.0.21. *Compact sets are closed.*

Proof. No Proof

□

Example 0.0.22. Note that $S = (0,1)$ in $\mathbb{R}$ is not compact. From the previous theorem, if S is compact it should be closed. Thus it is not compact although it is bounded.

Example 0.0.23. Note that $\mathbb{R}$ in $\mathbb{R}$ is not compact although it is closed. Because it is not bounded. Also, $S = [1, \infty)$ is closed but not compact.

So closed sets are not compact in general. Consider the next theorem.

Theorem 0.0.24. *A closed set S in a compact metric space (X, d) is compact*

Proof. Homework

□

Corollary 0.0.25. *If F is a closed in a compact set S in a metric space (X, d), then $F \cap S$ is compact.*

Proof. Homework

□

Theorem 0.0.26. *$S = [a, b]$ is compact in $\mathbb{R}$*

Proof. Suppose that S is not compact. Then there exists an open cover $\{G_\alpha\}$ that has no finite subcover. (We will try to find a contradiction).
Choose $c_1 \in S$ to split the interval in half. Then $\{G_\alpha\}$ covers both $[a, c_1]$ and $[c_1, b]$ and at least one of these intervals has no finite subcover say $I_1 = [a, c_1]$. Then subdivided again by choosing $c_2 \in [a, c_1]$, we get $[a, c_2]$, $[c_2, c_1]$ and at least one of them has no finite subcover, say $I_2 = [a, c_2]$. Continue, we obtain sequence

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

of nested close intervals, and non- of them has finite subcover of $\{G_\alpha\}$, also $|I_n| \to 0$ as $n \to \infty$.

By the nested intervals theorem there exists x in I_n for all n but each I_n is closed, thus x there exists G_{α_x} that contains x. But G_{α_x} is open, that is there exists $r > 0$ such that $B(x, r) \subseteq G_{\alpha_x}$. Since $|I_n| \to 0$ as $n \to \infty$, then some $I_n \subseteq B(x, r)$ meaning G_{α_x} covers I_n which a contradiction. Hence $[a, b]$ is compact. □

The previous theorem is also true in $\mathbb{R}^n$.

Theorem 0.0.27 (Hiene-Borel Theorem)**.** *In $\mathbb{R}$ (or $\mathbb{R}^n$) S is compact if and only if S is both closed and bounded.*

Proof.
($\Rightarrow$) Suppose that S is compact in $\mathbb{R}$ (or $\mathbb{R}^n$), then S is bounded and closed by Theorem 3.4.10 and Theorem 3.4.11.
($\Leftarrow$) (NOT TRUE IN ARBITRARY METRIC SPACE)

Let S be bounded and closed in $\mathbb{R}$. Then there exists $r > 0$ such that $S \subseteq (-r, r) \subseteq [-r, r]$. Since $[-r, r]$ is closed then $[-r, r]$ is compact by previous theorem. But S is closed, hence S is compact by Theorem 3.4.14. □

Example 0.0.28. *Discrete metric space on infinite set A. A is closed and bounded but not compact.*

Theorem 0.0.29. *Let (X, d) be a metric space. S is compact in X if and only if every infinite subset of S has a limit point in S.*

Corollary 0.0.30 (Bolzano-Weierstrass Theorem)**.**

Every bounded infinite subset of $\mathbb{R}^n$ has a limit point

Proof. Let E be a bounded infinite set in $\mathbb{R}^n$. Then there exists $x \in \mathbb{R}^n$ and $r > 0$ such that $E \subseteq B(x, r) \subseteq D(x, r)$. But $D(x, r)$ is closed and bounded in $\mathbb{R}^n$, hence by Hiene-Borel Theorem $D(x, r)$ is compact. Thus by previous theorem E has a limit point in $D(x, r)$. □

Definition 0.0.31 (Dense Sets)**.**

Let (X, d) be a metric space and let $S \subseteq X$. Then S is **dense** in X if and only if $\bar{S} = X$. That is for every $x \in X$ and for every $r > 0$ we have that $B(x, r) \cap S \neq \emptyset$, i.e., every open ball in (X, d) contains a point of S.

For example $\mathbb{Q}$ is dense in $\mathbb{R}$.

Chapter 1

Continuity

When we studied real-valued functions of a real variable in calculus, the techniques and theory built on properties of continuity, differentiability, and integrability. All of these concepts are defined using the precise idea of a limit. In this chapter, we want to look at continuous functions on metric spaces.

1.1 Definitions and Concepts

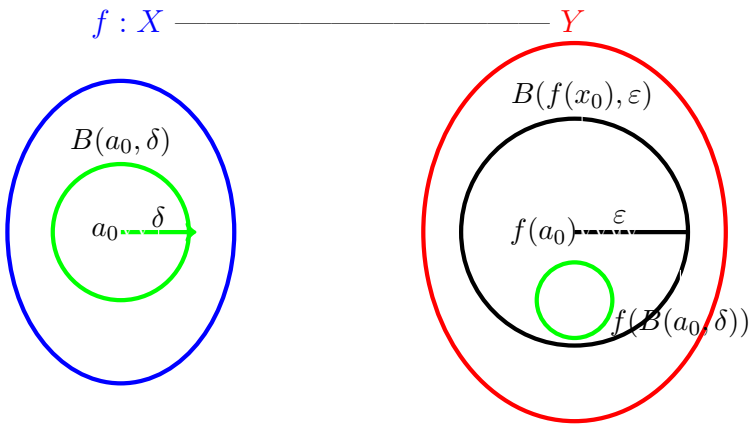
Recall that, for a function f to be continuous at a_0 we require Three things: f must be defined at a_0; $\lim_{x\to a_0} f(x)$ must be exist; and thirdly that the limit equal $f(a_0)$.

Definition 1.1.1. Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is **continuous** in $a_0 \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$ (depends on ε, a_0), such that

$$\text{if} \quad d_1(x, a_0) < \delta \qquad \text{then} \qquad d_2(f(x), f(a_0)) < \varepsilon, \qquad \text{for all } x \in X.$$

In other words, $f : X \to Y$ is **continuous** in $a_0 \in X$ if for each ball $B(f(a_0), \varepsilon) \in Y$ there exists $B(a_0, \delta) \in X$ such that

$$f(B(a_0, \delta)) \subseteq B(f(a_0), \varepsilon)$$



We say that f is **continuous** if it is continuous at every point of X.

If f is not continuous at $a_0 \in X$, we say that f is **discontinuous** at a_0.

Example 1.1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = x$, for all $x \in \mathbb{R}$. Then f is continuous.

Let $\varepsilon > 0$. Suppose that $a \in \mathbb{R}$, then

$$d(f(x), f(a)) = |x - a|$$

also

$$d(x, a) = |x - a|$$

thus there exists $\delta > 0$ with $\delta = \varepsilon$, such that

$$\text{if } d(x, a) = |x - a| < \delta \implies d(f(x), f(a)) = |x - a| < \varepsilon.$$

The next theorem characterizes continuous functions.

Theorem 1.1.3. *Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is continuous if and only if the inverse image of any open subset of Y is an open subset of X.*

That is, f is continuous if for any open set V in Y,

$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$

is open set in X.

Example 1.1.4. Show that the absolute function f on $\mathbb{R}$, i.e. $f(x) = |x|$ for all $x \in \mathbb{R}$ is continuous.

Solution: Suppose that $V = (a, b)$ is an open interval in $\mathbb{R}$, then

$$f^{-1}(V) := \begin{cases} \emptyset, & \text{if } a < b \leq 0, \\ (-b, b), & \text{if } a < 0 < b, \\ (-b, -a) \cup (a, b) & \text{if } 0 \leq a < b \end{cases}$$

we note that in each case $f^{-1}(V)$ is open in $\mathbb{R}$, thus f is continuous.

Example 1.1.5. Let X be any non-empty set. The discrete metric on X defined by

$$d(x, y) := \begin{cases} 0, & \text{when } x = y, \\ 1, & \text{when } x \neq y. \end{cases}$$

Show that every function $f : X \to X$, is continuous.

Solution: For any V open subset of Y, its inverse image $f^{-1}(V)$ is open subset of X, since every subset of a discrete metric space is open.

Theorem 1.1.6. *Let (X, d_1) and (Y, d_2) be metric spaces. A function $f : X \to Y$ is continuous if and only if the inverse image of any closed subset of Y is closed subset of X.*

That is, f is continuous if for any closed set S in Y,

$$f^{-1}(S) = \{x \in X : f(x) \in S\}$$

is closed set in X.

Proof. Homework, Hint: $f^{-1}(V^c) = (f^{-1}(V)^c)$

□

Example 1.1.7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, defined by

$$f(x) := \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1 & \text{if } x < 0 \end{cases}$$

Show that f is discontinuous at $x = 0$.

Solution: Suppose that $V = (\frac{-1}{2}, \frac{1}{2})$, then $f^{-1}(V) = \{0\}$. It is clear that f is not continuous at $x = 0$.

Prove that f is continuous for all $x > 0$.

Not that $f(x) = 1$ for all $x > 0$, thus if V is any open set in $\mathbb{R}$ that contains 1 then

$$f^{-1}(V) := \begin{cases} (0, \infty), & \text{if } -1, 0 \notin V, \\ [0, \infty), & \text{if } 0 \in V, -1 \notin V, \\ \mathbb{R} \setminus \{0\}, & \text{if } -1 \in V, 0 \notin V, \\ \mathbb{R} & \text{if } -1, 0 \in V. \end{cases}$$

In each case there exists open set U contains $x \in (0, \infty)$ such that

$$f(U) \subseteq V.$$

Example 1.1.8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = 2x - 1$. Prove that f is continuous for all $x \in \mathbb{R}$.

Solution: Suppose that $a_0 \in \mathbb{R}$ and $\varepsilon > 0$. Then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| = |2x - 2a_0| = 2|x - a_0|$$

also

$$d(x, a_0) = |x - a_0|$$

hence if we choose $\delta < \frac{\varepsilon}{2}$, then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| < \varepsilon, \qquad \text{for all } x \in \mathbb{R}.$$

Thus f is continuous function on $\mathbb{R}$.

Example 1.1.9. Let $f : (0, \infty) \to \mathbb{R}$ be a function defined by $f(x) = \sqrt{x}$. Prove that f is continuous for all $x \in (0, \infty)$.

Solution: Suppose that $a_0 \in (0, \infty)$ and $\varepsilon > 0$. Then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| = |\sqrt{x} - \sqrt{a_0}| = \left| \frac{x - a_0}{\sqrt{x} + \sqrt{a_0}} \right| \leq \frac{|x - a_0|}{\sqrt{a_0}}$$

also

$$d(x, a_0) = |x - a_0|$$

hence if we choose $\delta < \sqrt{a_0}\ \varepsilon$, then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| < \varepsilon, \qquad \text{for all } x \in \mathbb{R}.$$

Thus f is continuous function on $(0, \infty)$.

Homework

1. Use theorem 1.1.3 to prove the previous example.
2. Suppose that $f : [0, \infty) \to \mathbb{R}$ such that $f(x) = \sqrt{x}$, prove that f is continuous for all $x \in [0, \infty)$.

Example 1.1.10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^2$. Prove that f is continuous for all $x \in \mathbb{R}$.

Solution: Suppose that $a_0 \in \mathbb{R}$ and $\varepsilon > 0$. Then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| = \left|x^2 - a_0^2\right| = |(x - a_0)(x + a_0)|$$

$$= |x - a_0||x - a_0 + 2a_0| \leq |x - a_0|\left(|x - a_0| + 2|a_0|\right)$$

also

$$d(x, a_0) = |x - a_0|$$

hence if we choose $\delta \leq 1$ then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| < |x - a_0|\left(1 + 2|a_0|\right),$$

therefore if

$$\delta = \min\left\{1, \frac{\varepsilon}{1 + 2|a_0|}\right\}$$

then

$$d(f(x), f(a_0)) = |f(x) - f(a_0)| < \varepsilon.$$

Thus f is continuous function on $\mathbb{R}$.

The next theorem explains the relation between the continuity and convergent.

Theorem 1.1.11. *Let (X, d) be a metric space and $f : X \to X$ be a function. Suppose that $a_0 \in X$, then f is continuous at a_0 if and only if* ***for every convergent sequence*** $\{a_n\}_{n=1}^{\infty}$ ***to*** a_0 ***in*** X**,** $(a_n \to a_0)$**,** ***the sequence*** $\{f(a_n)\}_{n=1}^{\infty}$ ***converges to*** $f(a_0)$**,** $(f(a_n) \to f(a_0))$.

Example 1.1.12. Let $f : [a, b] \to \mathbb{R}$ be a function, defined by

$$f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 2 & \text{if } x \in \mathbb{Q}' \end{cases}$$

Show that f is discontinuous at all points $x \in [a, b]$.

Solution:

Let a_0 be a rational number, then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of irrational numbers such that

$$a_n \to a_0$$

also $f(a_n) = 2$ for all $n \in \mathbb{N}$ but $f(a_0) = 1$. Thus

$$f(a_n) \nrightarrow f(a_0)$$

hence using the previous theorem, f is discontinuous at a_0.

If a_0 is an irrational number, then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of rational numbers such that

$$a_n \to a_0$$

also $f(a_n) = 1$ for all $n \in \mathbb{N}$ but $f(a_0) = 2$. Thus

$$f(a_n) \nrightarrow f(a_0)$$

hence using the previous theorem, f is discontinuous at a_0.

Homework:

Use the definition to show that the function f in the previous example is discontinuous.

Theorem 1.1.13. *Let $(X, d_1), (Y, d_2)$ and (Z, d_3) be metric spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions. If f is continuous at $a_0 \in X$ and g is continuous at $f(a_0) \in Y$ then the function $g \circ f$ is continuous at a_0, where $g \circ f(x) = g(f(x))$*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in X such that $a_n \to a_0$. Since f is continuous at a_0 then

$$f(a_n) \to f(a_0) \qquad \text{(by theorem 1.1.11)}$$

also since g is continuous at $f(a_0)$ then

$$g(f(a_n)) \to g(f(a_0)) \qquad \text{(by theorem 1.1.11)}$$

that is

$$g \circ f(a_n) \to g \circ f(a_0)$$

thus $g \circ f$ is continuous at a_0 by theorem 1.1.11. □

1.2 Real Valued Functions-The space $C(X)$

In this section we will suppose that X is a metric space and we study the continuous functions $f : X \to \mathbb{R}$, (The continuous real valued functions). We will use $C(X)$ to represent this set. It is clear that $C(X)$ is non-empty set, because it it contains the constant function. The next theorem shows that $C(X)$ is a metric space.

Theorem 1.2.1. *Let f, g be continuous functions from X to $\mathbb{R}$. Then*

1. *$f + g$ is a continuous function where $(f + g)(x) = f(x) + g(x)$.*
2. *fg is a continuous function where $(fg)(x) = f(x)g(x)$.*
3. *For all $a \in \mathbb{R}$, the function af is continuous where $(af)(x) = af(x)$.*
4. *If $g(a_0) \neq 0$, then the function $\frac{f}{g}$ is continuous at a_0 where*

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

5. *The function $|f|$ is continuous where $|f|(x) = |f(x)|$.*

Proof. 1.

Let $a_0 \in X$ and $\{a_n\}_{n=1}^{\infty}$ be a sequence in X that converges to a_0, $(a_n \to a_0)$. Since f is continuous, then

$$f(a_n) \to f(a_0)$$

and also

$$g(a_n) \to g(a_0)$$

using theorem 2.3.4 we get

$$f(a_n) + g(a_n) \to f(a_0) + g(a_0)$$

that is

$$(f+g)(a_n) \to (f+g)(a_0)$$

thus $f+g$ is continuous at a_0. □

Proof. 2,3,4,5 Homework □

1.2.1 Real Continuous Functions Defined on Compact Spaces

Definition 1.2.2. Let $f : X \to \mathbb{R}$ be a function. Then f is called bounded if there exists $M > 0$ such that

$$f(x) \leq M, \qquad \text{for all } x \in X.$$

In other words, a function f is **bounded** if the range of f is bounded set in $\mathbb{R}$, where

$$\text{range of } f = \{f(x) : x \in X\}.$$

Lemma 1.2.3. *Let $(X,d),(Y,d_1)$ be metric spaces and $f : X \to Y$ be a continuous function. If X is a compact space then $f(X)$ is a compact set, where*

$$f(X) = \{f(x) : x \in X\}.$$

Theorem 1.2.4. *Let $f : X \to \mathbb{R}$ be a continuous function. If X is compact then f is bounded.*

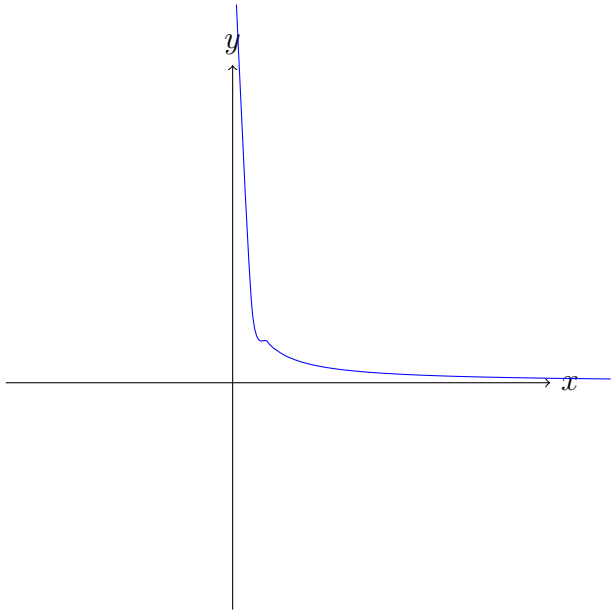
Remark 1.2.5.

1. If $f : X \to \mathbb{R}$ is a continuous function and X is not compact, then it is not necessary that f be bounded. For example

$$f(x) = \frac{1}{x}, \qquad x \in (0,\infty)$$

is continuous but not bounded. In particular for any $M > 0$ in $\mathbb{R}$ there exists $n \in \mathbb{N}$ such that

$$f(\frac{1}{n}) = n > M. \qquad \text{(Archimedean Property).}$$



2. It is possible that a function f be bounded and continuous however X is not compact. For example:

$$f(x) = 2x \qquad \text{for all } x \in (0, 1)$$

f is bonded and continuous, but $(0, 1)$ is not compact.

Definition 1.2.6. Let $f : X \to \mathbb{R}$ be a bounded function. Then f **has a global maximum point at** x_0 **if**

$$f(x_0) \geq f(x) \qquad \text{for all } x \in X.$$

Similarly, the function f **has a global minimum point at** y_0 **if**

$$f(y_0) \leq f(x) \qquad \text{for all } x \in X.$$

The value of the function at a maximum point ,$f(x_0)$, is called **the maximum value** of the function and the value of the function $f(y_0)$, at a minimum point is called the **minimum value of the function**.

Remark 1.2.7.

1. The global maximum point (if there exists) it may be not unique, also the global maximum point (if there exists) it may be not unique. However the maximum value (if there exists) it is unique and

the same thing for the minimum value. For example

$$f(x) = \sin x \qquad \text{for all } x \in \mathbb{R}.$$

has infinite number of global maximum points and global minimum points. In particular

$$f(\frac{\pi}{2} + 2n\pi) = 1 \qquad \text{for all } n \in \mathbb{N},$$

$$f(\frac{3\pi}{2} + 2n\pi) = -1 \qquad \text{for all } n \in \mathbb{N},$$

and

$$-1 \leq f(x) \leq 1 \qquad \text{for all } x \in \mathbb{R}.$$

2. It possible that a function f is bounded but does not has global maximum point (or global minimum point). For example

$$f(x) = 2x \qquad \text{for all } x \in (0, 1)$$

note that for all $x \in (0, 1)$, there exists x_1 such that

$$x < x_1 < 1 \qquad \text{and} \qquad f(x) < f(x_1).$$

Theorem 1.2.8. *Let $f : X \to \mathbb{R}$ be a continuous function. If X is compact space then there exists $x_0, y_0 \in X$ such that*

$$f(y_0) \leq f(x) \leq f(x_0), \qquad \textit{for all } x \in X.$$

1.3 The Uniform Continuity

Definition 1.3.1. Let (X, d) be a metric space. A function $f : X \to \mathbb{R}$ is called **uniformly continuous** on X if for every $\varepsilon > 0$ there exists $\delta > 0$ (δ depends on ε only) such that for every $x, y \in X$

$$\text{if} \quad d(x, y) < \delta \quad \Longrightarrow \text{then} \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Example 1.3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = 7x - 3$. Prove that f is uniformly continuous on $\mathbb{R}$.

Solution: Let $x, y \in \mathbb{R}$, and let $\varepsilon > 0$. Then

$$d(x, y) = |x - y|$$

and

$$|f(x) - f(y)| = 7|x - y|$$

thus if we choose $\delta = \frac{\varepsilon}{7}$ then we get

$$d(x, y) = |x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon \quad \text{for every } x, y \in \mathbb{R}.$$

Hence f is uniformly continuous on $\mathbb{R}$.

Note that every uniformly continuous function is continuous, however the converse is not true. Consider the following example:

Example 1.3.3. Let $f : (0, \infty) \to \mathbb{R}$ be a function defined by

$$f(x) = \frac{1}{x}$$

Show that f is not uniformly continuous on $(0, \infty)$.

Solution: Let $\varepsilon = 1$ we will show that for every $\delta > 0$ there exists $x, y \in (0, \infty)$ such that

$$|x - y| < \delta \quad \text{but} \quad |f(x) - f(y)| > 1.$$

Let $\delta > 0$ then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Suppose that $x, y \in (0, 1)$ such that $x = \frac{1}{n}$ and

$y = \frac{2}{n}$, then

$$|x - y| = \frac{1}{n} < \delta \quad \text{but} \quad |f(x) - f(y)| = \frac{n}{2} > 1.$$

Hence f is not uniformly continuous.

Example 1.3.4. Let $a > 0$ and let $f : [a, \infty) \to \mathbb{R}$ be a function such that

$$f(x) = \frac{1}{x} \quad \text{for all } x \in [a, \infty).$$

Prove that f is uniformly continuous on $[a, \infty)$.

Solution: Let $\varepsilon > 0$ and $x, y \in [a, \infty)$. Then

$$d(x, y) = |x - y|$$

and

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|y - x|}{xy} < \frac{|y - x|}{a^2},$$

thus if we choose $\delta = a^2\varepsilon$ then we get

$$d(x, y) = |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon \quad \text{for all } x, y \in [a, \infty).$$

Therefore f is uniformly continuous on $[a, \infty)$.

Theorem 1.3.5. *Let (X, d) be a compact metric space. If $f : X \to \mathbb{R}$ is continuous function on X then f is uniformly continuous on X.*

Corollary 1.3.6. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Then f is uniformly continuous on $[a, b]$.*

Chapter 2

Sequence of Functions

Let S be a subset of real numbers, $S \subseteq \mathbb{R}$. For each $n \in \mathbb{N}$ suppose that there exists a function $f_n : S \to \mathbb{R}$, then $\{f_n\}_{n=1}^{\infty}$ is a **sequence of functions** defined on S.

For each fixed $x \in S$, $f_n(x)$ is a sequence of numbers.

Example 2.0.1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on $\mathbb{R}$ as follows:

$$f_n(x) = \frac{x}{n} \quad \text{for all } x \in \mathbb{R} \text{ and for all } n \in \mathbb{N}.$$

Draw the graph of f_n for all $n \in \mathbb{N}$.

Solution:

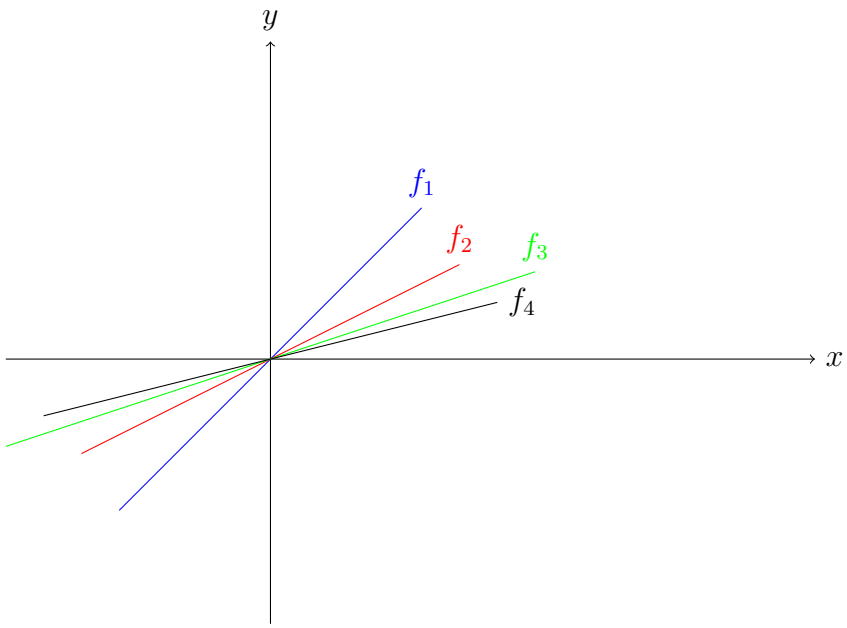
For $n = 1 \Longrightarrow f_1(x) = x$

For $n = 2 \Longrightarrow f_2(x) = \frac{x}{2}$

For $n = 3 \Longrightarrow f_3(x) = \frac{x}{3}$

For $n = 4 \Longrightarrow f_4(x) = \frac{x}{4}$

:



We notice that the sequence of numbers $\{f_n(x)\}_{n=1}^{\infty}$ converges to 0. **To prove that:**
Let $\varepsilon > 0$ then there exists $k \in \mathbb{N}$ such that $\frac{|x|}{k} < \varepsilon$ for all $x \in \mathbb{R}$. Thus

$$|f_n(x) - 0| = \left|\frac{x}{n}\right| < \frac{|x|}{k} < \varepsilon \quad \text{for all } n \geq k.$$

Now, if $f : \mathbb{R} \to \mathbb{R}$ is a function defined by $f(x) = 0$ for all $x \in \mathbb{R}$, then

$$f_n \longrightarrow f.$$

Example 2.0.2. Let $\{f_n\}_{n=1}^{n}$ be a sequence of functions defined on $\mathbb{R}$ as follows:

$$f_n(x) = \frac{nx}{1 + n^2x^2}, \quad \text{for all } x \in \mathbb{R} \text{ and for all } n \in \mathbb{N}.$$

Draw the graph of f_n for $n = 1, 2, 3$, and find the function f such that $f_n \to f$.

Solution:

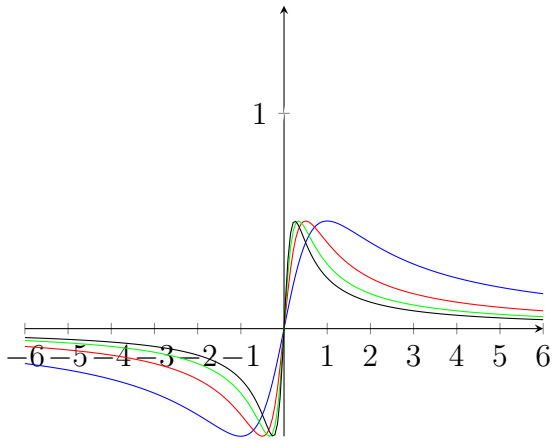
For $n = 1 \Longrightarrow f_1(x) = \frac{x}{1+x^2}$

For $n = 2 \Longrightarrow f_2(x) = \frac{2x}{1+4x^2}$

For $n = 3 \Longrightarrow f_3(x) = \frac{3x}{1+9x^2}$

For $n = 4 \Longrightarrow f_4(x) = \frac{4x}{1+16x^2}$

:



We notice that the sequence of numbers $\{f_n(x)\}_{n=1}^{\infty}$ converges to 0. If $x = 0$ then $f_n(0) = 0$ for all $n \in \mathbb{N}$, thus the sequence of numbers $\{f_n(0)\}_{n=1}^{\infty}$ converges to 0.
If $x \neq 0$, let $\varepsilon > 0$ then there exists $k \in \mathbb{N}$ such that $\frac{1}{k|x|} < \varepsilon$ for all $x \in S$. Thus

$$|f_n(x) - 0| = \left|\frac{nx}{1+n^2x^2}\right| < \left|\frac{nx}{n^2x^2}\right| = \frac{1}{n|x|} < \frac{1}{k|x|} < \varepsilon \quad \text{for all } n \geq k.$$

Now, if $f : \mathbb{R} \to \mathbb{R}$ is a function defined by $f(x) = 0$ for all $x \in \mathbb{R}$, then

$$f_n \longrightarrow f.$$

2.1 The pointwise convergence and the uniform convergence

Definition 2.1.1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on $S \subseteq \mathbb{R}$ and f be a function defined on S. Then The sequence $\{f_n\}_{n=1}^{\infty}$ **converges pointwise** to the function f and usually written as

$$f_n \longrightarrow f \quad \text{pointwise}$$

if and only the sequence of numbers $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$,

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all } x \in S,$$

that is for each $\varepsilon > 0$ and for each $x \in S$ there exists $k \in \mathbb{N}$ (k depends on ε, x), such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq k.$$

If there exists a fixed $k \in \mathbb{N}$ for all $x \in S$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in S \text{ and for all } n \in \mathbb{N},$$

then $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly to f,

$$f_n \longrightarrow f \quad \text{uniformly}$$

Example 2.1.2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be functions defined by

$$f_n(x) = \frac{\sin nx}{n}, \quad \text{for all } x \in \mathbb{R} \text{ and for all } n \in \mathbb{N}.$$

Draw the graph of f_n for $n = 1, 2, 3$, and find the function f such that $f_n \to f$ uniformly

Solution:

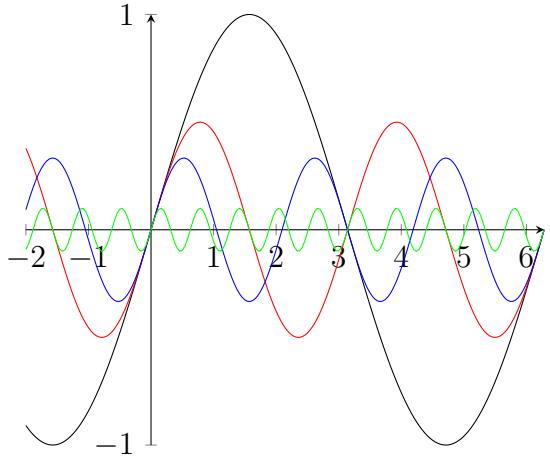
For $n = 1 \Longrightarrow f_1(x) = \sin x$

For $n = 2 \Longrightarrow f_2(x) = \frac{\sin 2x}{2}$

For $n = 3 \Longrightarrow f_3(x) = \frac{\sin 3x}{3}$

:

For $n = 10 \Longrightarrow f_{10}(x) = \frac{\sin 10x}{10}$



We notice that the sequence of numbers $\{f_n(x)\}_{n=1}^{\infty}$ converges to 0. To prove this:

let $\varepsilon > 0$ then there exists $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$, for all $n \geq k$ we have

$$|f_n(x) - 0| = \left|\frac{\sin nx}{n}\right| \leq \frac{1}{n} < \frac{1}{k} < \varepsilon$$

hence $f_n(x) \to 0$ pointwise. Since k hear does not depend on x then $f_n \to O$ uniformly.

Example 2.1.3. Let $f_n : [0,1] \to \mathbb{R}$ be functions defined by

$$f_n(x) = x^n, \quad \text{for all } x \in \mathbb{R} \text{ and for all } n \in \mathbb{N}.$$

Draw the graph of f_n for $n = 1, 2, 3$, and find the function f such that $f_n \to f$. Show that f_n does not converge uniformly to f.

Solution:

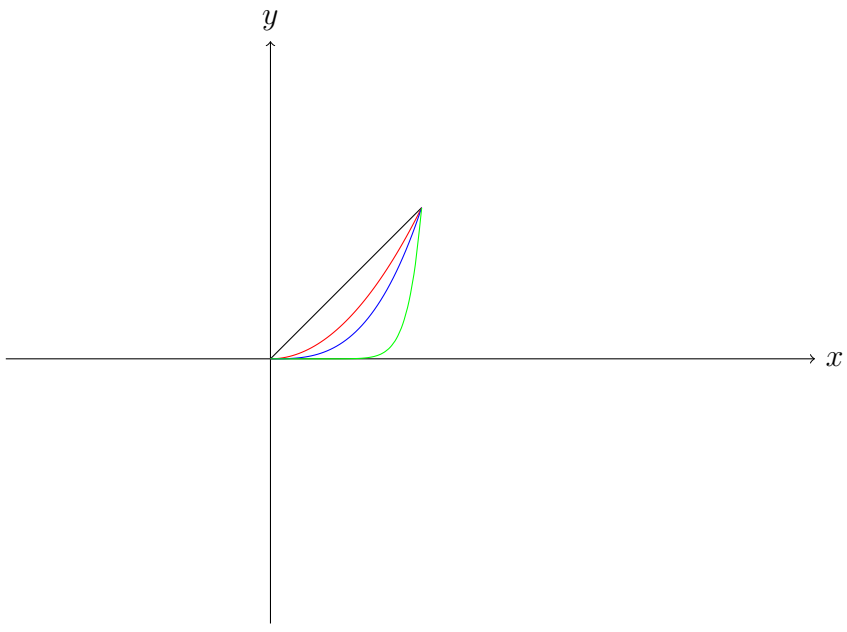
For $n = 1 \Longrightarrow f_1(x) = x$

For $n = 2 \Longrightarrow f_2(x) = x^2$

For $n = 3 \Longrightarrow f_3(x) = x^3$

:

For $n = 10 \Longrightarrow f_{10}(x) = x^{10}$



For each $0 < a < 1$ the sequence of numbers $\{a^n\}_{n=1}^{\infty}$ is decreasing and bounded below by 0, thus it is convergent. In particular for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $a^k < \varepsilon$, thus for all $n \geq k$ we have

$$|a^n - 0| = a^n < a^k < \varepsilon$$

hence $a^n \to 0$, for $a \in (0,1)$.

If $a = 0$ then it is clear that $a^n \to 0$.

If $a = 1$ then it is clear that $a^n \to 1$.

Therefore $f_n \to f$ where

$$f(x) := \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1 \end{cases}$$

Now, let $\varepsilon = \frac{1}{4}$ and let $x = \frac{1}{2}$, then $x^n = \left(\frac{1}{2}\right)^n$, for $n = 1$ we have

$$|x^n - 0| = \frac{1}{2} > \varepsilon = \frac{1}{4},$$

hence f_n does not converge uniformly to f.

2.2 Properties of Uniform Convergence

Theorem 2.2.1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on S. Suppose that f_n converges uniformly to the function f, then*

1. *If each f_n is bounded, then f is bounded.*

2. *If each f_n is continuous at $x_0 \in S$, then f is continuous at x_0. Therefore, if each f_n is continuous on S, then f is continuous on S*

Theorem 2.2.2. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions defined on S. Suppose that f_n converges to the continuous function f. If*

$$f_{n+1} \geq f_n \quad \text{or} \quad f_{n+1} \leq f_n \quad \text{for all } x \in S \quad \text{and} \quad \text{for all } n \in \mathbb{N},$$

and S is compact set, then f_n converges uniformly to f.

Theorem 2.2.3 (Weierstrass)**.**
For any continuous function $f : [a,b] \to \mathbb{R}$, there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ such that

$$P_n \longrightarrow f \quad \text{uniformly.}$$

Chapter 3

Riemann Integral

3.1 Definition of Integral

Definition 3.1.1. Let $J = [a, b]$ be a given finite interval. A **partition** $\mathcal{P}$ of $[a, b]$ is a finite set of points $\{x_0, x_1, x_2, \cdots, x_n\}$ satisfying $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. In other words

$$\mathcal{P} = \{a = x_0, x_1, x_2, \cdots, x_n = b : x_i < x_{i+1}, i = 0, 1, 2, \cdots, n-1\}.$$

The intervals

$$J_i = [x_{i-1}, x_i], \quad 1 \leq i \leq n,$$

are called **partition intervals**.

Notation: For two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$:

- $\mathcal{P}_1 \cup \mathcal{P}_2 \Rightarrow$ denote the partition formed from the points of $\mathcal{P}_1$ and $\mathcal{P}_2$.
- $\mathcal{P}_1 \subset \mathcal{P}_2 \Rightarrow$ means each point of $\mathcal{P}_1$ is a point of $\mathcal{P}_2$.

Definition 3.1.2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f has a **least upper bound** M and a **greatest lower bound** m on $J = [a, b]$:

$$M = \sup\{f(x) : x \in J\}, \quad \text{and} \quad m = \inf\{f(x) : x \in J\}.$$

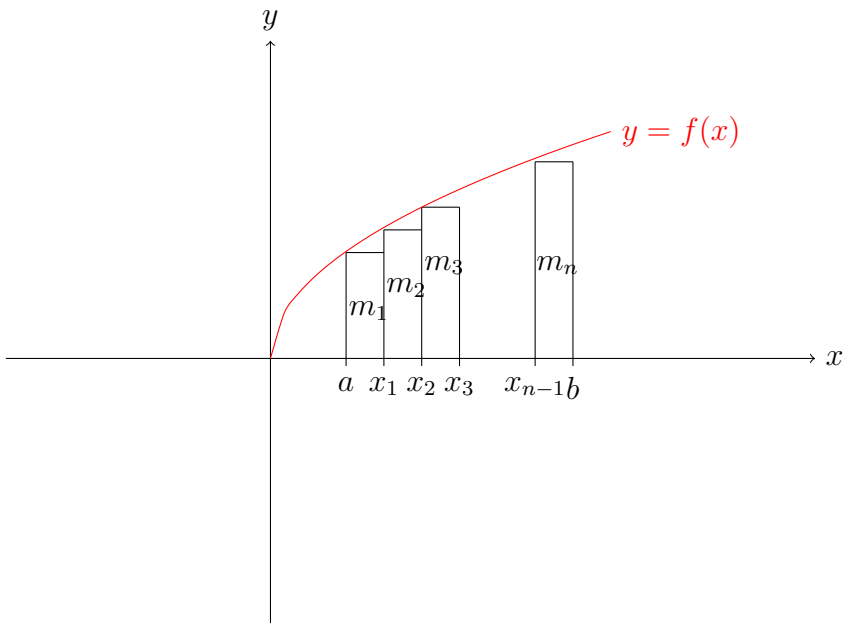
If $\mathcal{P}$ is a partition of $J = [a, b]$, then f is bounded on each subinterval $[x_{i-1}, x_i]$. Hence f has a **least**

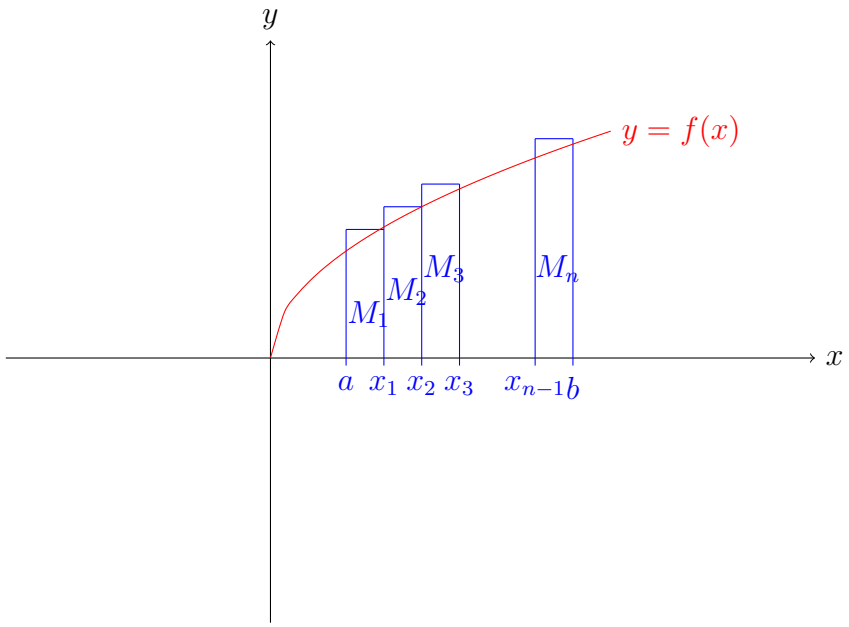
upper bound M_i and a **greatest lower bound** m_i on $J_i = [x_{i-1}, x_i]$:

$$M_i = \sup\{f(x) : x \in J_i\}, \quad \text{and} \quad m_i = \inf\{f(x) : x \in J_i\}.$$

It is clear that

$$m \leq m_i \leq M_i \leq M.$$





Definition 3.1.3.

The **Upper sum** of f relative to $\mathcal{P}$ is defined by

$$\bar{s}(f,\mathcal{P}) = \sum_{i=1}^{n} M_i|J_i| = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

The **Lower sum** of f relative to $\mathcal{P}$ is defined by

$$\underline{s}(f,\mathcal{P}) = \sum_{i=1}^{n} m_i|J_i| = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

Properties:

1. $\underline{s}(f,\mathcal{P}) \leq \bar{s}(f,\mathcal{P})$.

2. For any partitions $\mathcal{P}$ and $\mathcal{P}'$ with $\mathcal{P} \subset \mathcal{P}'$, we have

$$\underline{s}(f,\mathcal{P}) \leq \underline{s}(f,\mathcal{P}') \leq \bar{s}(f,\mathcal{P}') \leq \bar{s}(f,\mathcal{P}).$$

3. For any partitions $\mathcal{P}$ and $\mathcal{P}'$, $\underline{s}(f,\mathcal{P}) \leq \bar{s}(f,\mathcal{P}')$.

Definition 3.1.4.

Let

$$\bar{S} = \{\bar{s}(f,\mathcal{P}) : \mathcal{P} \text{ is a partiotion of } J\}$$

$$\underline{S} = \{\underline{s}(f,\mathcal{P}) : \mathcal{P} \text{ is a partiotion of } J\}$$

Now, if f is bounded then f is bounded above and bounded below on $[a,b]$. Thus there exists M and m with $m \leq f(x) \leq M$ for all $x \in [a,b]$. Hence each $\bar{S}$ and $\underline{S}$ is not empty, where $m(b-a) \in \underline{S}$ and $M(b-a) \in \bar{S}$. Therefore,

$$\overline{\int} f = \inf(\bar{S}) \quad \text{and} \quad \underline{\int} f = \sup(\underline{S})$$

exist,

where

$$\overline{\int} f = \inf(\bar{S}) \quad \text{is upper Riemann integral for } f \text{ on } [a,b]$$

$$\underline{\int} f = \sup(\underline{S}) \quad \text{is lower Riemann integral for } f \text{ on } [a,b]$$

.

Definition 3.1.5. A bounded function $f : [a, b] \to \mathbb{R}$ is called **Riemann integrable** if

$$\underline{\int} f = \overline{\int} f.$$

In which case, the common value of these two numbers is called the **Integral of** f and is denoted by

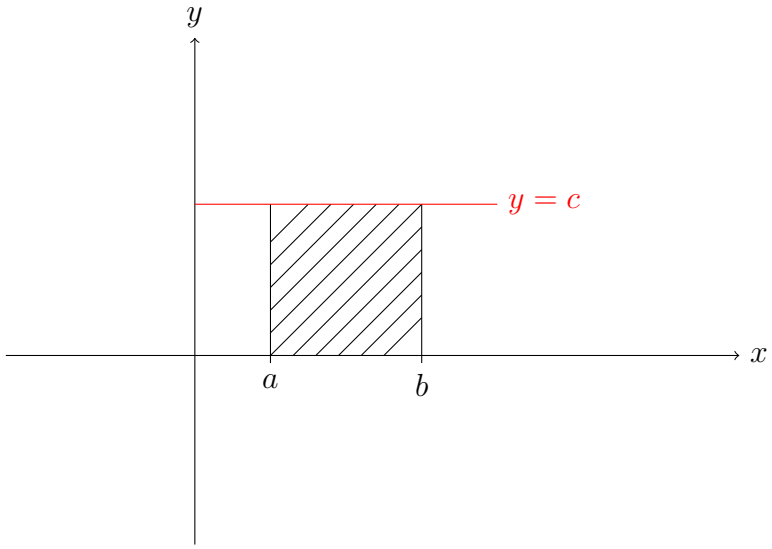
$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)dx.$$

Remark 3.1.6. Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Then

$$\int_b^a f = -\int_a^b f$$

Example 3.1.7. Let $f : [a, b] \to \mathbb{R}$ be the constant function with $f(x) = c$ for all $x \in [a, b]$. Prove that f is integrable function and $\int_a^b f = c(b - a)$.

Solution:



Let $\mathcal{P} = \{a = x_0, x_1, x_2, \cdots, x_n = b : x_{i-1} < x_i, i = 1, 2, \cdots, n\}$ be any partition on $[a, b]$.

$$\bar{s}(f, P) = \sum_{i=1}^{n} M_i|J_i| = \sum_{i=1}^{n} M_i|x_i - x_{i-1}| = \sum_{i=1}^{n} c|x_i - x_{i-1}|$$

$$= c\sum_{i=1}^{n} |x_i - x_{i-1}| = c(b - a).$$

$$\underline{s}(f,P) = \sum_{i=1}^{n} m_i|J_i| = \sum_{i=1}^{n} m_i|x_i - x_{i-1}| = \sum_{i=1}^{n} c|x_i - x_{i-1}|$$

$$= c\sum_{i=1}^{n} |x_i - x_{i-1}| = c(b-a).$$

$$\bar{S} = \{\bar{s}(f,P) : P \text{ is a partition on } [a,b]\} = \{c(b,a)\}$$

$$\underline{S} = \{\underline{s}(f,P) : P \text{ is a partition on } [a,b]\} = \{c(b,a)\}.$$

Thus

$$\overline{\int} f = \inf \bar{S} = c(b-a) \quad \text{and} \quad \underline{\int} f = \sup \underline{S} = c(b-a),$$

since $\overline{\int} f = \underline{\int} f$, then f is integrable on $[a,b]$ and $\int_a^b f = c(b-a)$.

Theorem 3.1.8 (Riemann's Condition)**.**

Let f be a bounded function on $[a,b]$. Then f is Riemann integrable on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition P on $[a,b]$ such that

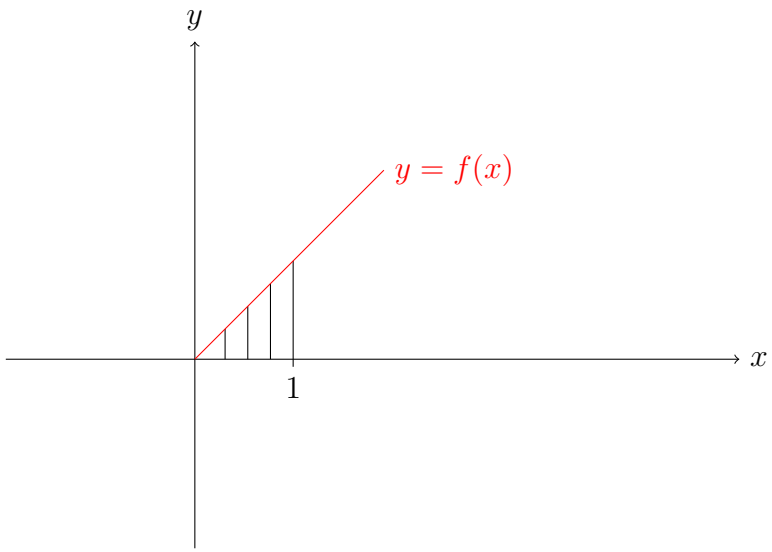
$$\bar{s}(f,P) - \underline{s}(f,P) < \varepsilon.$$

Proof. No Proof

□

Example 3.1.9. Let $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x$ for all $x \in [0,1]$. Prove that f is integrable and $\int_0^1 f = \frac{1}{2}$.

Solution:



Let $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \frac{4}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} = 1\right\}$ be a partition on $[0,1]$.

$$\bar{s}(f, P_n) = \sum_{i=1}^{n} M_i |J_i| = \sum_{i=1}^{n} M_i |x_i - x_{i-1}| = \sum_{i=1}^{n} \frac{i}{n} \left(\frac{1}{n}\right)$$

$$= \frac{1}{n^2} (1 + 2 + 3 + \cdots + n) = \frac{1}{n^2} \left(\frac{n(n+1)}{2}\right) = \frac{1}{2} + \frac{1}{2n},$$

$$\underline{s}(f, P_n) = \sum_{i=1}^{n} m_i |J_i| = \sum_{i=1}^{n} m_i |x_i - x_{i-1}| = \sum_{i=1}^{n} \frac{(i-1)}{n} \left(\frac{1}{n}\right)$$

$$= \frac{1}{n^2} (1 + 2 + 3 + \cdots + (n-1)) = \frac{1}{n^2} \left(\frac{(n-1)n}{2}\right) = \frac{1}{2} - \frac{1}{2n}.$$

Now, let $\varepsilon > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$, also

$$\bar{s}(f, P_n) - \underline{s}(f, P_n) = \frac{1}{n}, \text{ for all } n \in \mathbb{N}$$

thus $\bar{s}(f, P_n) - \underline{s}(f, P_n) < \varepsilon$. Using Riemann condition we get f is integrable.

Therefore $\overline{\int} f = \underline{\int} f$.

Note that

$$\bar{S} = \{\bar{s}(f, P_n) : P_n \text{ is a partition on } [a,b], \text{ for each } n \in \mathbb{N}\} = \left\{\frac{1}{2} + \frac{1}{2n}, n \in \mathbb{N}\right\},$$

$$\underline{S} = \{\underline{s}(f, P_n) : P_n \text{ is a partition on } [a,b], \text{ for each } n \in \mathbb{N}\} = \left\{\frac{1}{2} - \frac{1}{2n}, n \in \mathbb{N}\right\}.$$

We know that

$$\overline{\int} f = \inf \bar{S} = \inf \left\{\frac{1}{2} + \frac{1}{2n}\right\} \quad \text{and} \quad \underline{\int} f = \sup \underline{S} = \sup \left\{\frac{1}{2} - \frac{1}{2n}\right\},$$

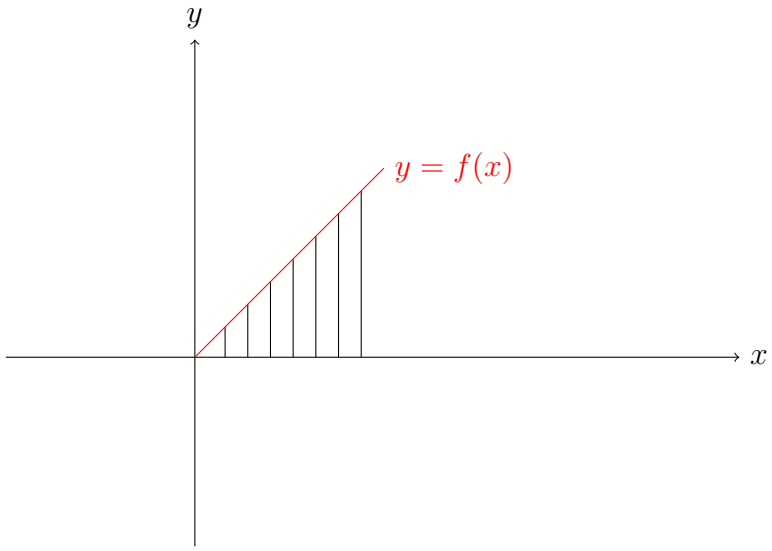
Since $\overline{\int} f = \inf \bar{S}$, which means that $\overline{\int} f \leq \left\{\frac{1}{2} + \frac{1}{2n}, \text{ for all } n \in \mathbb{N}\right\}$, then

$$\int_0^1 f = \frac{1}{2}.$$

QUIZ

Let $f : [0,3] \to \mathbb{R}$ defined by $f(x) = x$ for all $x \in [0,3]$. Prove that f is Riemann integrable and $\int_0^3 f = \frac{9}{2}$.

SOLUTION



Let $P_n = \left\{0, \frac{3}{n}, \frac{6}{n}, \frac{9}{n}, \frac{12}{n}, \ldots, \frac{3n}{n} = 3\right\}$ be a partition on $[0,3]$.

$$\bar{s}(f, P_n) = \sum_{i=1}^{n} M_i |J_i| = \sum_{i=1}^{n} M_i |x_i - x_{i-1}| = \sum_{i=1}^{n} \frac{3i}{n}\left(\frac{3}{n}\right)$$

$$= \frac{9}{n^2}(1 + 2 + 3 + \cdots + n) = \frac{9}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{9}{2} + \frac{9}{2n},$$

$$\underline{s}(f, P_n) = \sum_{i=1}^{n} m_i |J_i| = \sum_{i=1}^{n} m_i |x_i - x_{i-1}| = \sum_{i=1}^{n} \frac{3(i-1)}{n}\left(\frac{3}{n}\right)$$

$$= \frac{9}{n^2}(1 + 2 + 3 + \cdots + (n-1)) = \frac{9}{n^2}\left(\frac{(n-1)n}{2}\right) = \frac{9}{2} - \frac{9}{2n}.$$

Now, let $\varepsilon > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{9}{n} < \varepsilon$, also

$$\bar{s}(f, P_n) - \underline{s}(f, P_n) = \frac{9}{n}, \text{ for all } n \in \mathbb{N}$$

thus $\bar{s}(f, P_n) - \underline{s}(f, P_n) < \varepsilon$. Using Riemann condition we get f is integrable.

Therefore $\overline{\int} f = \underline{\int} f$.

Note that

$$\bar{S} = \{\bar{s}(f, P_n) : P_n \text{ is a partition on } [a,b], \text{ for each } n \in \mathbb{N}\} = \left\{\frac{9}{2} + \frac{9}{2n}, n \in \mathbb{N}\right\},$$

$$\underline{S} = \{\underline{s}(f, P_n) : P_n \text{ is a partition on } [a,b], \text{ for each } n \in \mathbb{N}\} = \left\{\frac{9}{2} - \frac{9}{2n}, n \in \mathbb{N}\right\}.$$

We know that

$$\overline{\int} f = \inf \bar{S} = \inf\left\{\frac{9}{2} + \frac{9}{2n}\right\} \quad \text{and} \quad \underline{\int} f = \sup \underline{S} = \sup\left\{\frac{9}{2} - \frac{9}{2n}\right\},$$

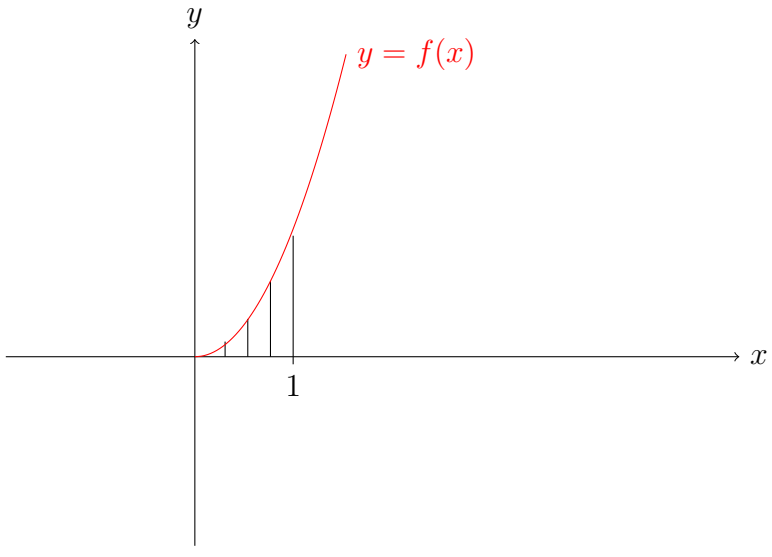
Since $\overline{\int} f = \inf \bar{S}$, which means that $\overline{\int} f \leq \left\{\frac{9}{2} + \frac{9}{2n}, \text{ for all } n \in \mathbb{N}\right\}$, then

$$\int_0^1 f = \frac{9}{2}.$$

HOMEWORK

Let $f : [0,1] \to \mathbb{R}$ be a function defined by $f(x) = x^2$ for all $x \in [0,1]$. Show that f is integrable and find $\int_0^1 f$.

Solution:



Let $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \frac{4}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} = 1\right\}$ be a partition on $[0,1]$.

$$\bar{s}(f, P_n) = \sum_{i=1}^{n} M_i |J_i| = \sum_{i=1}^{n} M_i |x_i - x_{i-1}| = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right)$$

$$= \frac{1}{n^3}\left(1+4+9+\cdots+n^2\right) = \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2},$$

$$\underline{s}(f,P_n) = \sum_{i=1}^{n} m_i|J_i| = \sum_{i=1}^{n} m_i|x_i - x_{i-1}| = \sum_{i=1}^{n}\left(\frac{(i-1)}{n}\right)^2\left(\frac{1}{n}\right)$$

$$= \frac{1}{n^2}\left(1+4+9+\cdots+(n-1)^2\right) = \frac{1}{n^3}\left(\frac{(n-1)n(2n-1)}{6}\right) = \frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}.$$

Now, let $\varepsilon > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$, also

$$\bar{s}(f,P_n) - \underline{s}(f,P_n) = \frac{1}{n}, \text{ for all } n \in \mathbb{N}$$

thus $\bar{s}(f,P_n) - \underline{s}(f,P_n) < \varepsilon$. Using Riemann condition we get f is integrable.

$$\overline{\int} f = \inf \bar{S} = \inf\left\{\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right\} \Longrightarrow \overline{\int} f \leq \left\{\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right\} \quad \text{for all } n \in \mathbb{N},$$

$$\Longrightarrow \overline{\int} f \leq \frac{1}{3}$$

and

$$\underline{\int} f = \sup \underline{S} = \sup\left\{\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}\right\} \Longrightarrow \underline{\int} f \geq \left\{\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}\right\} \quad \text{for all } n \in \mathbb{N},$$

$$\Longrightarrow \underline{\int} f \geq \frac{1}{3}$$

Therefore,

$$\overline{\int} f \leq \frac{1}{3} \leq \underline{\int} f,$$

since f is integrable then $\overline{\int} f = \underline{\int} f$, this implies that

$$\int_0^1 f(x)dx = \frac{1}{3}.$$

Example 3.1.10. Prove that

$$f(x) := \begin{cases} 5 & \text{if } x \in \mathbb{Q}, \\ -3 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable on any interval $[a, b]$.

Solution:

Example 3.1.11. Prove that

$$f(x) := \begin{cases} 3 & \text{if } x \leq 0, \\ 7 & \text{if } x > 0 \end{cases}$$

is Riemann integrable on $[-5, 5]$.

Solution:

Let $I_n = \left[-\frac{1}{n}, \frac{1}{n}\right]$, $A_n = \left[-5, -\frac{1}{n}\right]$, $B_n = \left[\frac{1}{n}, 5\right]$. Suppose that

$P_n^1 = \left\{-5 = x_0, x_1, x_2, \cdots, x_k = -\frac{1}{n}\right\}$ be a partition on A_n, and

$P_n^2 = \left\{\frac{1}{n} = x_0, x_1, x_2, \cdots, x_m = 5\right\}$ be a partition on B_n, then

$P_n = P_n^1 \cup I_n \cup P_n^2$ be a partition on $[-5, 5]$.

It is clear that

$$\bar{s}(f, P_n) = 3\left(5 - \frac{1}{n}\right) + 7\left(\frac{2}{n}\right) + 7\left(5 - \frac{1}{n}\right) = 50 + \frac{4}{n},$$

and

$$\underline{s}(f, P_n) = 3\left(5 - \frac{1}{n}\right) + 3\left(\frac{2}{n}\right) + 7\left(5 - \frac{1}{n}\right) = 50 - \frac{4}{n}.$$

Let $\varepsilon > 0$ then there exists $n \in \mathbb{N}$ such that $\frac{8}{n} < \varepsilon$ hence

$$\bar{s}(f, P_n) - \underline{s}(f, P_n) = \frac{8}{n} < \varepsilon,$$

thus using the Riemann's condition we get f is integrable function on $[-5, 5]$.
To find $\int_{-5}^{5} f$, we know that

$$\overline{\int} f = \inf \bar{S} = \inf\left\{50 + \frac{4}{n}\right\} \Longrightarrow \overline{\int} f \leq \left\{50 + \frac{4}{n}\right\} \quad \text{for all } n \in \mathbb{N},$$

$$\Longrightarrow \overline{\int} f \leq 50$$

and

$$\underline{\int} f = \sup \underline{S} = \sup\left\{50 - \frac{4}{n}\right\} \Longrightarrow \overline{\int} f \leq \left\{50 - \frac{4}{n}\right\} \quad \text{for all } n \in \mathbb{N},$$

$$\Longrightarrow \underline{\int} f \geq 50 \qquad .$$

Therefore,

$$\overline{\int} f \leq 50 \leq \underline{\int} f,$$

since f is integrable then $\overline{\int} f = \underline{\int} f$, thus

$$\int_{-5}^{5} f(x)dx = 50.$$

3.2 Continuous Functions and Integration

Recall that

• Every continuous function f defined on a closed interval $J = [a, b]$ is bounded and has global maximum point and has minimum global point. (Theorem 1.2.8), thus

$$M = \max\{f(x) : x \in J\} \quad \text{and} \quad m = \min\{f(x) : x \in J\}$$

• Every continuous function f defined on a closed interval $J = [a, b]$ is uniformly continuous. (Theorem 1.3.5), that is for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in [a, b]$

$$\text{if} \quad |x - y| < \delta \implies \text{then} \implies |f(x) - f(y)| < \varepsilon.$$

Theorem 3.2.1. *Let f be a continuous function on $[a, b]$. Then f is Riemann integrable on $[a, b]$.*

Proof. No Proof

□

3.3 Monotone Functions

Definition 3.3.1 (Monotone Function)**.**
Let X be a non-empty set and $f : X \to \mathbb{R}$ be a function. Then:

- f is called **increasing** if for every $x, y \in X$ and $x < y$ then $f(x) \leq f(y)$.
- f is called **strictly increasing** if for every $x, y \in X$ and $x < y$ then $f(x) < f(y)$ for every $x, y \in X$.
- f is called **decreasing** if for every $x, y \in X$ and $x \leq y$ then $f(x) \geq f(y)$.
- f is called **strictly decreasing** if for every $x, y \in X$ and $x < y$ then $f(x) > f(y)$.

Remark 3.3.2.
1. If f is continuous then f may not be monotone. For example Let $f : [-2, 2] \to \mathbb{R}$ be a function such that $f(x) = x^2$ for all $x \in [-2, 2]$. Then f is continuous but not monotone.
2. If f is monotone then f may not be continuous. For example Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = [x]$. Then f is monotone but not continuous.
3. Every monotone function f defined on closed interval $[a, b]$ is bounded. In particular if f is monotone increasing on $[a, b]$ the $f(b)$ is the maximum value of f and $f(a)$ is the minimum value of f. If f is monotone decreasing on $[a, b]$ the $f(b)$ is the minimum value of f and $f(a)$ is the maximum value of f.

Theorem 3.3.3. *Let $f : [a, b] \to \mathbb{R}$ be a monotone function. Then f is Riemann integrable function.*

Proof. No Proof

□

3.4 Bounded Riemann Integrable Functions

Definition 3.4.1 (Negligible Set)**.**

Let S be a non-empty set of real numbers. Then S is called **negligible set or (zero set)** if there exists a countable collection of open interval $\{I_k\}$ such that

1. $S \subseteq \bigcup_k I_k$

2. $\sum_{k=1} |I_k| < \varepsilon, \quad$ for every $\varepsilon > 0$.

Theorem 3.4.2.

1. *Every countable (finite or infinite) set of real numbers is negligible.*

2. *Every subset of negligible set is negligible.*

3. *The union of any countable (finite or infinite) collection of negligible set is negligible.*

Remarks and Examples

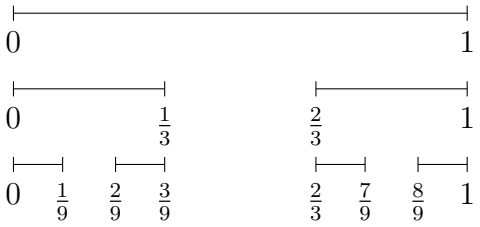
1. If $I = [a, b]$ then I is not negligible.

2. The set of rational numbers $\mathbb{Q}$ is negligible.

3. The set of irrational numbers $\mathbb{Q}'$ is not negligible.

4. There exists a negligible set which is uncountable. For example Cantor set:

Start with the interval $[0, 1]$. remove the (open) middle third of it, i.e. get $\left(\frac{1}{3}, \frac{2}{3}\right)$. Now remove the middle thirds of each of the remaining intervals, i.e. get $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$. Continue this process to infinity. The points left over form a fractal called the **standard Cantor set**. It is an infinite set since a lot of points, including the endpoints of the removed intervals, are never removed.



Theorem 3.4.3 (Lebesgue Theorem in Riemann's Integral)**.**
Let $f : [a, b] \to \mathbb{R}$ be bounded function. Suppose that D is the set of all points in $[a, b]$ where the function f is not continuous. Then f is Riemann integrable if and only if D is negligible set.

Corollary 3.4.4. *Let $f : [a, b] \to \mathbb{R}$ be a monotone function on $[a, b]$. Then the set D of all points in $[a, b]$ where the function f is not continuous is negligible.*

Corollary 3.4.5. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If f is integrable on $[a, b]$ and $J \subseteq [a, b]$ then f is integrable on J.*

Corollary 3.4.6. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $c \in [a, b]$. If f is integrable on $[a, c]$ and $[c, b]$ then f is integrable on $[a, b]$.*

3.5 The Linear Space of Integrable Functions

Let $B[a, b]$ be the set of all real valued bounded functions defined on $[a, b]$. It is clear that if $f, g \in B[a, b]$ and r is a real number then $f + g$ and rf are belong $B[a, b]$, thus $B[a, b]$ is a vector space. Moreover, $fg \in B[a, b]$.

Let $C[a, b]$ be the space of all continuous functions defined on $[a, b]$. It is known that $C[a, b]$ is a subspace of $B[a, b]$.

Let $RI[a, b]$ be the space of all bounded integrable functions defined on $[a, b]$. The next theorem shows that $RI[a, b]$ is a subspace of $B[a, b]$

Theorem 3.5.1. *The set $RI[a, b]$ a subspace of $B[a, b]$. That is*

1. $rf \in RI[a,b]$ *for all* $f \in RI[a,b]$ *and* $r \in \mathbb{R}$.
2. $f+g \in RI[a,b]$ *for all* $f,g \in RI[a,b]$.

Theorem 3.5.2. *Let* $RI[a,b]$ *be the set of all bounded integrable function defined on* $[a,b]$. *Then*

1. $fg \in RI[a,b]$ *for all* $f,g \in RI[a,b]$.
2. *If* $\frac{1}{f}$ *is bounded then* $\frac{1}{f} \in RI[a,b]$ *for all* $f \in RI[a,b]$.

Theorem 3.5.3 (The Integration is a Linear Transformation)**.**

The function

$$\int : RI[a,b] \to \mathbb{R}$$

is a linear transformation, that is for all $f,g \in RI[a,b]$ *and* $r \in \mathbb{R}$

1. $\int rf = r\int f$
2. $\int (f+g) = \int f + \int g$.

Now, does the function $\int$ is monotone? consider the following theorem.

Theorem 3.5.4 (The monotonicity of Integration)**.**

Let $f,g : [a,b] \to \mathbb{R}$ *be bounded functions. Then*

1. *If* $f(x) \geq 0$ *for all* $x \in [a,b]$ *then*
$$\int_a^b f \geq 0$$
2. *If* $f(x) \geq g(x)$ *for all* $x \in [a,b]$ *then*
$$\int_a^b f \geq \int_a^b g$$

Proof. No Proof □

Corollary 3.5.5. *If* $f \in RI[a,b]$ *then* $|f| \in RI[a,b]$ *and*

$$\left|\int_a^b f\right| < \int_a^b |f|$$

3.6 Continuity of Integration

We noticed that the integration is a linear function and it is a monotone function. In this section we investigate the continuity of integration.

Example 3.6.1.

It is known that the set of all rational numbers in each interval $[a, b]$ is countable set, thus we can write its elements as $\{r_1, r_2, \cdots, r_n, \cdots\}$.

For each $n \in \mathbb{N}$ let $f_n : [a, b] \to \mathbb{R}$ be a sequence of functions defined by:

$$f_n(x) := \begin{cases} -3 & \text{if } x \in \{r_1, r_2, \cdots, r_n\}, \\ 2 & \text{if } x \notin \{r_1, r_2, \cdots, r_n\} \end{cases}$$

Then f_n is continuous at each points in $[a, b]$ except the set $\{r_1, r_2, \cdots, r_n\}$.

Since this set (the set where the function f_n is discontinuous) is finite set, then

f_n is integrable and $\int_a^b f_n = 2(b - a)$,

Let $f : [a, b] \to \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} -3 & \text{if x is rational number in } [a, b], \\ 2 & \text{if x is irrational number in } [a, b] \end{cases}$$

Then $f_n \to f$.

however f is not integrable.

Example 3.6.2. For each $n \in \mathbb{N}$ let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of functions defined by

$$f_n(x) := \begin{cases} 2nx & 0 \le x \le \frac{1}{2n}, \\ 2 - 2nx & \frac{1}{2n} \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} \le x \le 1 \end{cases}$$

Then:

- Draw the graph of f_n for $n = 1, 2, 3$
- Show that f_n is continuous for each n.

Thus f_n is integrable.

- Prove that $\int_0^1 f_n = 1$.

The sequence $f_n \to f$ where f is the zero function. $f(x) = 0$ for all $x \in [0, 1]$ is integrable function, however

$$\int_0^1 f_n \nrightarrow f$$

Theorem 3.6.3. *Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of bounded Riemann integrable functions. Suppose that f_n converges uniformly on $[a,b]$ to the function f. Then*

1. *f is Riemann integrable.*

2. *$\int_a^b f_n \to \int_a^b f$.*

Proof. Suppose that $f_n : [a,b] \to \mathbb{R}$ is bounded Riemann integrable function for each $n \in \mathbb{N}$. Then since f_n converges to f uniformly and f_n is bounded for all $n \in \mathbb{N}$ then f is bounded on $[a,b]$ (Theorem 2.2.1).

For each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2(b-a)} \quad \text{for all } n > k \text{ and for all } x \in [a,b].$$

Since each f_n is Riemann integrable function then the set D_n (the set of all points in [a,b] where f_n is discontinuous) is negligible set (Theorem 3.4.3).

Let $D = \bigcup_n D_n$, then D is negligible set.

Let $D' = [a,b] \backslash D_n$, then each f_n is continuous on D'.

Thus f is continuous on D' (Theorem 2.2.1).

Therefore if $D(f)$ is the set of all points in $[a,b]$ where f is discontinuous, then $D(f) \subseteq D$, that is $D(f)$ is negligible set. This implies that f is Riemann integrable function (Theorem 3.4.3).

Now for all $n > k$ we have

$$\left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b (f_n - f) \right| \leq \int_a^b |f_n - f| = \frac{\varepsilon}{2(b-a)}(b-a)$$

hence

$$\left| \int_a^b f_n - \int_a^b f \right| < \varepsilon$$

thus

$$\int_a^b f_n \longrightarrow \int_a^b f$$

□

Chapter 4

Differentiation Real-Valued Functions

4.1 Definitions and Examples

Definition 4.1.1. Let $f : (a, b) \to \mathbb{R}$ be a function, and $x_0 \in (a, b)$. We say that the function f is differentiable at x_0, if there exists a real number α (α depends on x_0), such that for each sequence x_n, ($x_n \neq x_0$ for all n) converges to x_0, the sequence

$$\frac{f(x_n) - f(x_0)}{x_n - x_0}$$

converges to α.

That is, f is differentiable at x_0 if there exists $\alpha \in \mathbb{R}$ such that

$$\text{for each } \; x_n \longrightarrow x_0 \quad (x_n \neq x_0, \forall n), \qquad \frac{f(x_n) - f(x_0)}{x_n - x_0} \longrightarrow \alpha.$$

α is called the differential of f at x_0 and denoted by

$$\frac{df}{dx}|_{x_0} \quad or \quad Df|_{x_0} \;\; or \;\; \alpha = f^{'}(x_0).$$

Remark 4.1.2.

- *If α exists, then it is unique (since the limit of convergent sequence is unique).*
- *If f is differentiable at each $x \in (a, b)$, then we say that f is differentiable on (a, b).*

Example 4.1.3.

Let $f : (a, b) \to \mathbb{R}$ be a function defined by $f(x) = c$ for all $x \in (a, b)$. For $x_0 \in (a, b)$, prove that

$f^{'}(x_0) = 0$.

SOLUTION:

Let $x_0 \in (a, b)$ and let x_n be a sequence in (a, b) such that $x_n \to x_0$ with ($x_n \neq x_0$ for all $n \in \mathbb{N}$). Then

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \frac{c - c}{x_n - x_0} \longrightarrow 0 = \alpha.$$

Thus f is differentiable at $x_0 \in (a, b)$ and $f^{'}(x_0) = 0$.

Example 4.1.4.

Let $f : (a, b) \to \mathbb{R}$ be a function defined by $f(x) = x$ for all $x \in (a, b)$. For $x_0 \in (a, b)$, prove that $f^{'}(x_0) = 1$.

SOLUTION:

Let $x_0 \in (a, b)$ and let x_n be a sequence in (a, b) such that $x_n \to x_0$ with ($x_n \neq x_0$ for all $n \in \mathbb{N}$). Then

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \frac{x_n - x_0}{x_n - x_0} \longrightarrow 1.$$

Thus f is differentiable at $x_0 \in (a, b)$ and $f^{'}(x_0) = 1$.

Example 4.1.5.

Let $f : (a, b) \to \mathbb{R}$ be a function defined by $f(x) = x^2$ for all $x \in (a, b)$. For $x_0 \in (a, b)$, prove that $f^{'}(x_0) = 2x_0$.

SOLUTION:

Let $x_0 \in (a, b)$ and let x_n be a sequence in (a, b) such that $x_n \to x_0$ with ($x_n \neq x_0$ for all $n \in \mathbb{N}$). Then

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \frac{x_n^2 - x_0^2}{x_n - x_0} = \frac{(x_n - x_0)(x_n + x_0)}{(x_n - x_0)} = x_n + x_0,$$

but $x_n \to x_0$ as $n \to \infty$, hence

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} \longrightarrow 2x_0.$$

Thus f is differentiable at $x_0 \in (a, b)$ and $f^{'}(x_0) = 2x_0$.

Example 4.1.6.

Let $f : (a, b) \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x}$ for all $x \in (a, b)$, $x \neq 0$. For $x_0 \in (a, b)$, prove that $f^{'}(x_0) = \frac{-1}{x_0^2}$.

SOLUTION:

Let $x_0 \in (a, b)$ and let x_n be a sequence in (a, b) such that $x_n \to x_0$ with ($x_n \neq x_0$ for all $n \in \mathbb{N}$). Then

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \frac{\frac{1}{x_n} - \frac{1}{x_0}}{x_n - x_0} = \frac{\frac{x_0 - x_n}{x_0 x_n}}{(x_n - x_0)} = \frac{(x_0 - x_n)}{x_0 x_n}.\frac{-1}{(x_0 - x_n)} = \frac{-1}{x_0 x_n},$$

but $x_n \to x_0$ as $n \to \infty$, hence

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} \longrightarrow \frac{-1}{x_0^2}.$$

Thus f is differentiable at $x_0 \in (a, b)$ and $f^{'}(x_0) = \frac{-1}{x_0^2}$.

Example 4.1.7.

Let $f : [0, \infty) \to \mathbb{R}$ be a function defined by $f(x) = \sqrt{x}$ for all $x \in [0, \infty)$. For $x_0 \in [0, \infty)$, prove that $f^{'}(x_0) = \frac{1}{2\sqrt{x_0}}$.

SOLUTION: HOMEWORK

Example 4.1.8.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by:

$$f(x) := \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ x & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that f is not differentiable at $x = 0$.

SOLUTION: Let x_n be a sequence in $\mathbb{R}$ such that $x_n \to 0$ (with $x_n \neq 0$, for all $n \in \mathbb{N}$). Then if x_n is a sequence of rational numbers, then

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{0 - 0}{x_n - 0} \longrightarrow 0$$

if x_n is a sequence of irrational numbers, then

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{x_n - 0}{x_n - 0} \longrightarrow 1$$

but $1 \neq 0$, thus f is not differentiable at $x = 0$.

4.2 Relation between differentiation and continuity

Theorem 4.2.1.

Let $f : (a, b) \to \mathbb{R}$ be a function. If f is differentiable at $x_0 \in (a, b)$ then f is continuous at x_0

Proof. Let f be a function that differentiable at x_0.

Define a function $w(x)$ as follows:

$$w(x) := \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} - \alpha, & \text{if } x \neq x_0 \\ 0, & \text{if } x = x_0. \end{cases}$$

We will show that the function f is continuous.

Let x_n be a sequence such that $x_n \to x_0$ with ($x_n \neq x_0$, for all $n \in \mathbb{N}$). Then

$$w(x_n) = \frac{f(x_n) - f(x_0)}{x_n - x_0} - \alpha,$$

since f is differentiable at x_0, then $\frac{f(x_n)-f(x_0)}{x_n-x_0} \longrightarrow \alpha$, thus $w(x_n) \longrightarrow 0$.
This means that $w(x_n) \longrightarrow w(x_0) = 0$. Therefore, w is continuous at x_0.

Now, from the definition of w we have

$$w(x) = \frac{f(x) - f(x_0)}{x - x_0} - \alpha \Longrightarrow w(x)(x - x_0) = f(x) - f(x_0) - \alpha(x - x_0),$$

so

$$f(x) = f(x_0) + \alpha(x - x_0) + w(x)(x - x_0)$$

note that, all functions on the right hand side are continuous, thus f is continuous at x_0. □

The converse of the previous theorem is not true in general. Consider the following example.

Example 4.2.2.

Let $f : (-2, 2) \to \mathbb{R}$ be a function defined by $f(x) = |x|$. Then f is not differentiable at $x_0 = 0$. However f is continuous at $x_0 = 0$ and in particular, f is continuous for all $x \in (-2, 2)$.

SOLUTION:

Let x_n be a sequence given by $x_n := \frac{1}{n}, n = 1, 2, 3, \cdots$. Then it is clear that $\frac{1}{n} \longrightarrow 0$ and

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{\frac{1}{n} - 0}{\frac{1}{n} - 0} \longrightarrow 1.$$

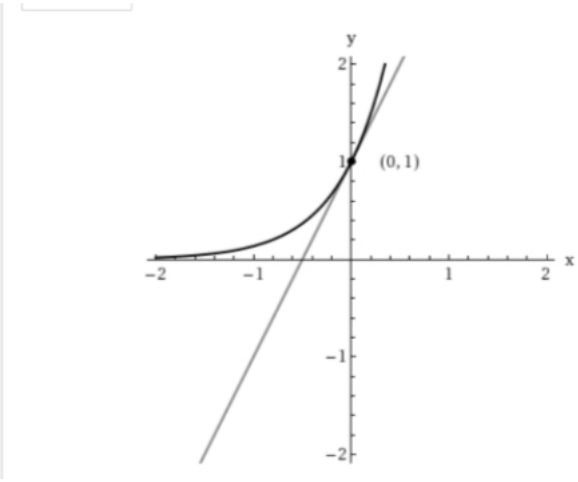
Now, let y_n be a sequence given by $y_n := \frac{-1}{n}, n = 1, 2, 3, \cdots$. Then it is clear that $\frac{-1}{n} \longrightarrow 0$ and

$$\frac{f(y_n) - f(0)}{y_n - 0} = \frac{\frac{1}{n} - 0}{\frac{-1}{n} - 0} \longrightarrow -1.$$

but $1 \neq -1$, hence, by definition f is not differentiable at $x_0 = 0$.

Remark 4.2.3.

Let $f : (a, b) \to \mathbb{R}$ be a function that differentiable at $x_0 \in (a, b)$. Then f can be approximated by a straight line that pass through the point $(x_0, f(x_0))$. This line called the tangent of the curve of f and it's slope is the differentiation of f at x_0.



Theorem 4.2.4.

Let I be an open interval and let f, g be real valued functions defined on I. If f and g are differentiable at $x_0 \in I$, then:

(1) *The function $f + g$ is differentiable at x_0 and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.*

(2) *The function fg is differentiable at x_0 and $(fg)'(x_0) = f(x_0)g'(x_0) + g(x_0)f'(x_0)$.*

(3) *The function cf, (c is a constant) is differentiable at x_0 and $(cf)'(x_0) = cf'(x_0)$.*

(4) *If $g(x_0) \neq 0$, then the function $\frac{f}{g}$ is differentiable at x_0 and*

$$(\frac{f}{g})^{'}(x_0) = \frac{g(x_0)f^{'}(x_0) - f(x_0)g^{'}(x_0)}{[g(x_0)]^2}.$$

Proof (1). Let x_n be a sequence in I such that $x_n \longrightarrow x_0$, with ($x_n \neq x_0$, for all $x \in \mathbb{N}$). Then

$$\frac{(f+g)(x_n) - (f+g)(x_0)}{x_n - x_0} = \frac{(f(x_n) + g(x_n)) - (f(x_0) + g(x_0))}{x_n - x_0}$$

$$= \frac{f(x_n) - f(x_0)}{x_n - x_0} + \frac{g(x_n) - g(x_0)}{x_n - x_0}.$$

Since f is differentiable at x_0, then $\frac{f(x_n)-f(x_0)}{x_n-x_0} \longrightarrow f^{'}(x_0)$,
and since g is differentiable at x_0, then $\frac{g(x_n)-g(x_0)}{x_n-x_0} \longrightarrow g^{'}(x_0)$,

thus

$$\frac{(f+g)(x_n) - (f+g)(x_0)}{x_n - x_0} \longrightarrow f^{'}(x_0) + g^{'}(x_0).$$

That is $f+g$ is differentiable at x_0 and $(f+g)^{'}(x_0) = f^{'}(x_0) + g^{'}(x_0)$.

Proof (2). Let x_n be a sequence in I such that $x_n \longrightarrow x_0$, with ($x_n \neq x_0$, for all $x \in \mathbb{N}$). Then

$$\frac{(fg)(x_n) - (fg)(x_0)}{x_n - x_0} = \frac{f(x_n)g(x_n) - f(x_0)g(x_0)}{x_n - x_0}$$

$$= \frac{f(x_n)g(x_n) - f(x_n)g(x_0) + f(x_n)g(x_0) - f(x_0)g(x_0)}{x_n - x_0}$$

$$= \frac{f(x_n)\left[g(x_n) - g(x_0)\right] + g(x_0)\left[f(x_n) - f(x_0)\right]}{x_n - x_0}$$

$$= f(x_n)\frac{g(x_n) - g(x_0)}{x_n - x_0} + g(x_0)\frac{f(x_n) - f(x_0)}{x_n - x_0}$$

Since f is differentiable at x_0, then $\frac{f(x_n)-f(x_0)}{x_n-x_0} \longrightarrow f^{'}(x_0)$,
and since g is differentiable at x_0, then $\frac{g(x_n)-g(x_0)}{x_n-x_0} \longrightarrow g^{'}(x_0)$,
also, since f is continuous at x_0 (Theorem 4.2.1), the $f(x_n) \longrightarrow f(x_0)$.
Thus

$$\frac{(fg)(x_n) - (fg)(x_0)}{x_n - x_0} \longrightarrow f(x_0)g^{'}(x_0) + g(x_0)f^{'}(x_0).$$

That is fg is differentiable at x_0 and $(fg)^{'}(x_0) = f(x_0)g^{'}(x_0) + g(x_0)f^{'}(x_0)$.

Proof (3). HOMEWORK

Proof (4). No Proof.

4.3 Differentiation and Integration

Example 4.3.1.

Let $f : [1, 2] \to \mathbb{R}$ be a function defined by $f(x) = x^2$, for all $x \in [1, 2]$. Then

$$\int_1^2 x^2 dx = \frac{x^3}{3}|_1^2 = \frac{7}{3}$$

we notice that

$$\frac{d}{dx}\int_1^2 x^2 dx = \frac{d}{dx}\left(\frac{7}{3}\right) = 0 \neq f(x) = x^2.$$

Example 4.3.2.

$$\int_1^t x^2 dx = \frac{x^3}{3}|_1^t = \frac{t^3}{3} - \frac{1}{3}$$

we notice that

$$\frac{d}{dt}\int_1^t x^2 dx = \frac{d}{dt}\left(\frac{t^3}{3} - \frac{1}{3}\right) = t^2.$$

Theorem 4.3.3 (The Fundamental Theorem on Calculus)**.**

Let $f : [a, b] \to \mathbb{R}$ be a bounded function on $[a, b]$ and integrable on $[a, b]$. Define

$$F(x) = \begin{cases} \int_a^x f, & a < x \leq b \\ 0, & x = a. \end{cases}$$

Then:

- *F is uniformly continuous function.*
- *If f is continuous then F is differentiable on (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$.*

  That is F is anti-derivative of f.

Corollary 4.3.4.

Let f be a real valued continuous function on $[a, b]$. If F is anti-derivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$